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COMMENTS ON SOLOW "STRUCTURE OF LINEAR MODELS"

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Comments on Solow "Structure of Linear Models"

I. N. Herstein*

In his paper "On the Structure of Linear Models" Solow makes extensive use of some results about the characteristic roots of indecomposable nonnegative matrices which are due to Frobenius [2]. A recent paper by Helmut Wielandt [5] gives a drastically simplified proof of these results of Frobenius; moreover, in several places the results are sharpened. One of the purposes of writing this note is to bring this paper and its translation into English by Herstein [3] to the attention of economists. Debreu and Herstein [1] have succeeded in simplifying even further the proof of these results.

It is pointed out here how these results are applicable in either simplifying, or obviating the need of, several of Solow's proofs. ( )

Wielandt's two theorems, in essence, are:

Theorem 1. Let A be a nonnegative, indecomposable matrix. Then A has a positive characteristic root r so that

a) r is a simple root

b) if α is any characteristic root of A, $|\alpha| \leq r$.

c) to r can be associated a positive characteristic vector.

Theorem 2. Let A be a nonnegative, indecomposable matrix. Suppose further that A has k roots of absolute value r (see Theorem 1). Then these roots are all simple and of the form $re^{\frac{2\pi i}{k}\lambda}$, $\lambda = 1, 2, \ldots, k$. Moreover a permutation matrix P exists so that PAP^{-1} has the form

$$\begin{bmatrix} 0 & A_{12} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & A_{23} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & A_{k-1,k} \\ A_{k1} & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

with square submatrices on the main diagonal.

A is said to be primitive if k = 1. By Theorem 2 it is clear that the concept of primitivity is equivalent to that of Solow's cyclicity (Solow, Theorem 2). In addition, Theorem 2 yields that every indecomposable, nonnegative matrix which is not primitive is cyclic in Solow's sense; the converse to this is trivially true. (Solow does not explicitly point out, in his definition of cyclicity, that the submatrices on the main diagonal should be square). Solow's w now has the interesting interpretation as k, the number of characteristic roots of largest absolute value. If A has a positive diagonal element, this diagonal element must remain on the diagonal of PAP^{-1} for every permutation matrix P, so Theorem 2 gives that A must be primitive (Solow, corollary to Theorem 2). It might be appropriate to point out the following here: If A is indecomposable and nonnegative, then A raised to some power is positive if and only if A is primitive [5. p. 647].

We give below a very simple proof of Solow's Theorem 1.

Theorem (Solow) If A is a nonnegative and indecomposable matrix whose row sums are never greater than 1 and at least one of whose row sums is less than one, then all the characteristic roots of A have absolute value less than one.

We use the following result established easily in [1] : let A be nonnegative and indecomposable, let r(A) be the largest positive characteristic root. Then if any element of A is increased, r(A) is also increased.[1/] It is enough by Theorem 1 to show that $r(A) < 1$.

Assume that the i^{th} row sum is less than 1. Let A_1 be constructed from A by enlarging only a_{ii} so that the sum of the i^{th} row of A_1 is not greater than 1. Then $r(A) < r(A_1)$, and since it is trivial that $r(A_1) \leq 1$, the theorem is established.

Cowles Commission for Research in Economics

References

[1] Debreu, Gerard, and Herstein, I. N., "Nonnegative Matrices," Cowles Commission Discussion Paper, Mathematics, No. 414, March 4, 1952.

[2] Frobenius, G., "Uber Matrizen aus nicht negativen Elementen," Sitzungsberichte Preussichen Akademie der Wissenschaften, Berlin, 1912, p. 456-477.

[3] Herstein, I. N., "Indecomposable, Nonnegative Matrices," Cowles Commission Discussion Paper, Mathematics, No. 413, Febru[illegible].

[4] Solow, Robert, "On the Structure of Linear [illegible]. Vol. 20, January 1952, p. 29-46.

[5] Wielandt, H., "Unzerlegbare, [illegible] Mathematische Zeitschrift, 1950, [illegible] p. 642-[illegible]

• [illegible] was carried out as part of a Cowles Commission project [illegible] contract with The RAND Corporation.

1. Footnote:
As has been kindly pointed out to me by Solow, this result by itself has the following economic significance: increasing any marginal propensity to import has a destabilizing effect on a system of the kind considered in his paper.